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HAAR WAVELET ANALYSIS OF CLIMATIC TIME SERIES

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In order to extract the intrinsic information of climatic time series from background red noise, in this paper, we will first give an analytic formula on the distribution of Haar wavelet power spectra of red noise in a rigorous statistical framework. After that, by comparing the difference of wavelet power spectra of real climatic time series and red noise, we can extract intrinsic features of climatic time series. Finally, we use our method to analyze Arctic Oscillation (AO) which is a key aspect of climate variability in the Northern Hemisphere, and discover a great change in fundamental properties of the AO, commonly called a regime shift or tripping point.

Keywords: Climatic time series; Haar wavelet analysis; red noise.

AMS Subject Classification: 42C

1. Introduction

The continuous wavelet transform possesses the ability to construct a time-frequency representation of a time series that offers very good time and frequency localization, so wavelet transforms can analyze localized intermittent periodicities of climatic time series.[1,7,8] The relative resolution in time and frequency space depends on the wavelet chosen. Many climatic time series have been analyzed with the Morlet wavelet.[2,3,5] which is roughly equally localized in time and frequency. However, the Haar wavelet which we discuss has not been previously used in this respect.

Physical interpretation of climatic time series requires statistical significant testing against a null hypothesis of some noise model. Different from regular signal analysis, climatic time series analysis often use red noise as noise model.[4] A simple model for red noise is the univariate lag-1 autoregressive (AR(1)) process. By comparing the difference of wavelet power spectra of real climatic series and AR(1) red noise, one can extract the intrinsic features of climatic time series. To date significance testing has been based on an empirical formula for the Morlet wavelet power spectra of AR(1) red noise.[2,3,5]

In this paper, we will focus on Haar wavelet. We will give an analytic formula of the distribution of Haar wavelet power spectra for an AR(1) red noise in a rigorous statistical framework (see Sec. 2). The relation between scale a and Fourier period T for the Morlet wavelet is $a = 0.97T$.[5] However, for Haar wavelet, the corresponding formula is $a = 0.3710T$ (see Appendix A). Since for any time series of time step δt and total length $N\delta t$, the range of scales is from the smallest resolvable scale $2\delta t$ to the largest scale $N\delta t$ in wavelet-based time series analysis, by using the Haar wavelet analysis, one can extract more low frequency intrinsic information. At the end of this paper, we use Haar wavelet to do the statistical significance test of the Arctic Oscillation (AO) and discover a great change in fundamental properties of the AO, commonly called a regime shift or tripping point.

2. Haar Wavelet Power Spectrum Distribution for AR(1) Time Series

Different from regular signal analysis, climatic time series analysis often use red noise as noise model.[4] A simple model for red noise is the univariate lag-1 autoregressive (AR(1)) process. A discrete random process $\{x_n\}_0^{N-1}$ is called an AR(1) time series with the parameter α if

$$x_n = \alpha x_{n-1} + \mathbb{Z}_n, \quad n = 1, \ldots, N-1 \quad \text{and} \quad x_0 = 0,$$

where the parameter $0 \le \alpha < 1$ and $\{\mathbb{Z}_n\}_1^\infty$ is the white noise, i.e. they are independent normal distributed variables with variance σ^2. From this, we can successively obtain that

$$x_n = \sum_{k=1}^{n} \alpha^{n-k} Z_k \tag{2.1}$$

and the expectation

$$E[\,x_n\,] = \sum_{k=1}^{n} \alpha^{n-k} E[\,Z_k\,] = 0.$$

Therefore, each x_n is a Gaussian random variable with mean 0. Let $H(x)$ be the Haar function:

$$H(x) = \begin{cases} 1, & 0 < x \le \frac{1}{2}, \\ -1, & -\frac{1}{2} < x < 0, \\ 0, & \text{otherwise}. \end{cases} \tag{2.2}$$

The wavelet transform $W_\nu(s)$ of AR(1) time series $\{x_n\}_0^{N-1}$ associated with $H(x)$ is defined ordinarily as[2,3]:

$$\sqrt{\frac{s}{\delta t}}\, W_\nu(s) = \sum_{n=0}^{N-1} x_n\, H\left(\frac{(n-\nu)\delta t}{s}\right), \tag{2.3}$$

where ν is the time parameter and s is the scaling parameter and δt is a sample period of $\{x_n\}_0^{N-1}$. By (2.2),

$$H\left(\frac{(n-\nu)\delta t}{s}\right) = \begin{cases} 1 & \left(\nu < n \le \nu + \frac{s}{2\delta t}\right), \\ -1 & \left(\nu - \frac{s}{2\delta t} < n < \nu\right), \\ 0 & \left(n = \nu \text{ or } n \le \nu - \frac{s}{2\delta t} \text{ or } n > \nu + \frac{s}{2\delta t}\right). \end{cases} \tag{2.4}$$

We choose ν, s such that

$$\frac{s}{2\delta t} \le \nu \le N - 1 - \frac{s}{2\delta t}. \tag{2.5}$$

By (2.3),

$$\sqrt{\frac{s}{\delta t}} E[W_\nu(s)] = \sum_{n=0}^{N-1} E[x_n] H\left(\frac{(n-\nu)\delta t}{s}\right) = 0.$$

Therefore, $W_\nu(s)$ is a Gaussian random variable with mean 0. To obtain the distribution of the power spectrum $|W_\nu(s)|^2$, we only need to compute the variance of $W_\nu(s)$. Note that

$$\mathrm{Var}(W_\nu(s)) = E[(W_\nu(s))^2] - (E[W_\nu(s)])^2 = E[(W_\nu(s))^2].$$

From this and (2.3), we have

$$\begin{aligned} \frac{s}{\delta t}\mathrm{Var}(W_\nu(s)) &= E\left[\left(\sum_{n=0}^{N-1} x_n H\left(\frac{(n-\nu)\delta t}{s}\right)\right)^2\right] \\ &= \sum_{n,m=0}^{N-1} E[x_n x_m] H\left(\frac{(n-\nu)\delta t}{s}\right) H\left(\frac{(m-\nu)\delta t}{s}\right). \end{aligned} \tag{2.6}$$

Denote $s^* = \frac{s}{2\delta t}$. By (2.5), we have

$$\nu - s^* \geq 0 \quad \text{and} \quad \nu + s^* \leq N - 1.$$

From this and (2.4), by (2.6), we get

$$2s^* \mathrm{Var}(W_\nu(s)) = \sum_{n,m=\nu-s^*}^{\nu+s^*} E[x_n x_m] H\left(\frac{m-\nu}{2s^*}\right) H\left(\frac{n-\nu}{2s^*}\right). \tag{2.7}$$

Since $\{Z_n\}_1^\infty$ is white noise with variance σ^2,

$$E[Z_k Z_l] = 0 (k \neq l) \quad \text{and} \quad E[Z_k^2] = \sigma^2.$$

By (2.1), we get

$$\begin{aligned} E[x_n x_m] &= E\left[\left(\sum_{k=1}^{m} \alpha^{m-k} Z_k\right)\left(\sum_{k=1}^{n} \alpha^{n-k} Z_k\right)\right] \\ &= \sum_{k=1}^{\lambda_{mn}} \alpha^{n-k} \alpha^{m-k} E[Z_k^2] \\ &= \sigma^2 \alpha^{n+m} \sum_{k=1}^{\lambda_{mn}} \alpha^{-2k} = \frac{\sigma^2 \alpha^{n+m}(1 - \alpha^{-2\lambda_{mn}})}{\alpha^2 - 1}, \end{aligned}$$

where $\lambda_{mn} = \min\{n, m\}$. From this, we get that

$$\left(\frac{\alpha^2 - 1}{\sigma^2}\right) E[x_n x_m] = \begin{cases} \alpha^{m+n} - \alpha^{n-m}, & m \leq n, \\ \alpha^{m+n} - \alpha^{m-n}, & m > n. \end{cases} \tag{2.8}$$

So, we get

$$\left(\frac{\alpha^2 - 1}{\sigma^2}\right) E[\, x_n x_m \,] = \alpha^{m+n} - \alpha^{|m-n|}.$$

From this and (2.7), we have

$$\begin{aligned} -2s^* \left(\frac{\alpha^2 - 1}{\sigma^2}\right) \mathrm{Var}(W_\nu(s)) &= \sum_{n,m=\nu-s^*}^{\nu+s^*} (\alpha^{|m-n|} - \alpha^{m+n}) H\left(\frac{m-\nu}{2s^*}\right) \\ &\quad \times H\left(\frac{n-\nu}{2s^*}\right) = J - \widetilde{J}, \end{aligned} \tag{2.9}$$

where

$$J = \sum_{n,m=\nu-s^*}^{\nu+s^*} \alpha^{|m-n|} H\left(\frac{m-\nu}{2s^*}\right) H\left(\frac{n-\nu}{2s^*}\right)$$

and

$$\tilde{J} = \sum_{n,m=\nu-s^*}^{\nu+s^*} \alpha^{m+n} H\left(\frac{m-\nu}{2s^*}\right) H\left(\frac{n-\nu}{2s^*}\right). \tag{2.10}$$

1 We first compute J.

By (2.4), we divide J into four sums:

$$\begin{aligned} J &= \left(\sum_{n,m=\nu-s^*}^{\nu-1} - \sum_{m=\nu-s^*}^{\nu-1} \sum_{n=\nu+1}^{\nu+s^*} - \sum_{m=\nu+1}^{\nu+s^*} \sum_{n=\nu-s^*}^{\nu-1} + \sum_{n,m=\nu+1}^{\nu+s^*} \right) \alpha^{|m-n|} \\ &=: J_1 - J_2 - J_3 + J_4. \end{aligned} \tag{2.11}$$

We compute each sum. For J_1:

$$J_1 = \left(\sum_{m=\nu-s^*}^{\nu-1} \sum_{n=\nu-s^*}^{m} + \sum_{m=\nu-s^*}^{\nu-1} \sum_{n=m+1}^{\nu-1} \right) \alpha^{|m-n|} = J_{11} + J_{12},$$

where

$$\begin{aligned} J_{11} &= \sum_{m=\nu-s^*}^{\nu-1} \alpha^m \left(\sum_{n=\nu-s^*}^{m} \alpha^{-n} \right) \\ &= \sum_{m=\nu-s^*}^{\nu-1} \frac{\alpha^m(\alpha^{-\nu+s^*} - \alpha^{-m-1})}{1-\alpha^{-1}} = \frac{\alpha^{-\nu+s^*}}{1-\alpha^{-1}} \sum_{m=\nu-s^*}^{\nu-1} \alpha^m - \sum_{m=\nu-s^*}^{\nu-1} \frac{\alpha^{-1}}{1-\alpha^{-1}} \\ &= -\frac{(\alpha^{-\nu+s^*})(\alpha^{\nu} - \alpha^{\nu-s^*})}{(1-\alpha)(1-\alpha^{-1})} - \frac{s^*}{\alpha(1-\alpha^{-1})} = \frac{1-\alpha^{s^*}}{(1-\alpha^{-1})(1-\alpha)} + \frac{s^*}{1-\alpha} \end{aligned}$$

and

$$\begin{aligned} J_{12} &= \sum_{m=\nu-s^*}^{\nu-1} \alpha^{-m} \left(\sum_{n=m+1}^{\nu-1} \alpha^{n} \right) \\ &= \sum_{m=\nu-s^*}^{\nu-1} \frac{\alpha^{-m}(\alpha^{m+1} - \alpha^{\nu})}{1-\alpha} = -\frac{\alpha^{\nu}}{1-\alpha} \sum_{m=\nu-s^*}^{\nu-1} \alpha^{-m} + \frac{1}{1-\alpha} \sum_{m=\nu-s^*}^{\nu-1} \alpha \\ &= -\frac{\alpha^{\nu}(\alpha^{-\nu+s^*} - \alpha^{-\nu})}{(1-\alpha)(1-\alpha^{-1})} + \frac{\alpha s^*}{1-\alpha} = \frac{1-\alpha^{s^*}}{(1-\alpha)(1-\alpha^{-1})} + \frac{\alpha s^*}{1-\alpha}. \end{aligned}$$

Therefore, we have

$$J_1 = J_{11} + J_{12} = \frac{2(1-\alpha^{s^*})}{(1-\alpha)(1-\alpha^{-1})} + \frac{(1+\alpha)s^*}{1-\alpha}.$$

For J_4:

$$J_4 = \sum_{m=\nu+1}^{\nu+s^*} \left(\sum_{n=\nu+1}^{m} + \sum_{n=m+1}^{\nu+s^*} \right) \alpha^{|m-n|} = J_{41} + J_{42},$$

where

$$J_{41} = \sum_{m=\nu+1}^{\nu+s^*} \alpha^m \left(\sum_{n=\nu+1}^{m} \alpha^{-n} \right)$$

$$= \sum_{m=\nu+1}^{\nu+s^*} \frac{\alpha^m(\alpha^{-\nu-1} - \alpha^{-m-1})}{1-\alpha^{-1}} = \frac{\alpha^{-\nu-1}}{1-\alpha^{-1}} \sum_{m=\nu+1}^{\nu+s^*} \alpha^m - \sum_{m=\nu+1}^{\nu+s^*} \frac{\alpha^{-1}}{1-\alpha^{-1}}$$

$$= \frac{\alpha^{-\nu-1}(\alpha^{\nu+1} - \alpha^{\nu+s^*+1})}{(1-\alpha)(1-\alpha^{-1})} - \frac{s^*}{\alpha(1-\alpha^{-1})} = \frac{1-\alpha^{s^*}}{(1-\alpha)(1-\alpha^{-1})} + \frac{s^*}{1-\alpha}$$

and

$$J_{42} = \sum_{m=\nu+1}^{\nu+s^*} \alpha^{-m} \left(\sum_{n=m+1}^{\nu+s^*} \alpha^n \right)$$

$$= \sum_{m=\nu+1}^{\nu+s^*} \frac{\alpha^{-m}(\alpha^{m+1} - \alpha^{\nu+s^*+1})}{1-\alpha} = -\frac{\alpha^{\nu+s^*+1}}{1-\alpha} \sum_{m=\nu+1}^{\nu+s^*} \alpha^{-m} + \sum_{m=\nu+1}^{\nu+s^*} \frac{\alpha}{1-\alpha}$$

$$= -\frac{\alpha^{\nu+s^*+1}(\alpha^{-\nu-1} - \alpha^{-\nu-s^*-1})}{(1-\alpha)(1-\alpha^{-1})} + \frac{\alpha s^*}{1-\alpha} = -\frac{\alpha^{s^*} - 1}{(1-\alpha)(1-\alpha^{-1})} + \frac{\alpha s^*}{1-\alpha},$$

it follows that

$$J_4 = J_{41} + J_{42} = \frac{(1+\alpha)s^*}{1-\alpha} + \frac{2(1-\alpha^{s^*})}{(1-\alpha)(1-\alpha^{-1})}.$$

Furthermore,

$$J_1 + J_4 = \frac{4(1-\alpha^{s^*})}{(1-\alpha)(1-\alpha^{-1})} + \frac{2(1+\alpha)s^*}{1-\alpha}.$$

Finally, for J_2 and J_3:

$$J_2 = \left(\sum_{m=\nu-s^*}^{\nu-1} \alpha^{-m} \right) \left(\sum_{n=\nu+1}^{\nu+s^*} \alpha^n \right)$$

$$= \left(\frac{\alpha^{-\nu+s^*} - \alpha^{-\nu}}{1-\alpha^{-1}} \right) \left(\frac{\alpha^{\nu+1} - \alpha^{\nu+s^*+1}}{1-\alpha} \right) = -\frac{\alpha(\alpha^{s^*} - 1)^2}{(1-\alpha)(1-\alpha^{-1})}$$

and

$$J_3 = \left(\sum_{m=\nu+1}^{\nu+s^*} \alpha^m \right) \left(\sum_{m=\nu-s^*}^{\nu-1} \alpha^{-n} \right) = J_2.$$

We obtain by (2.11) that

$$J = J_1 - J_2 - J_3 + J_4$$

$$= \frac{4(1-\alpha^{s^*})}{(1-\alpha)(1-\alpha^{-1})} + \frac{2\alpha(\alpha^{s^*} - 1)^2}{(1-\alpha)(1-\alpha^{-1})} + \frac{2s^*(1+\alpha)}{1-\alpha}$$

$$= \frac{2s^*(1-\alpha^2) - 4\alpha(1-\alpha^{s^*}) - 2\alpha^2(1-\alpha^{s^*})^2}{(1-\alpha)^2} =: \sigma_H^2. \qquad (2.12)$$

Next, we compute $\widetilde{J}$. By (2.4) and (2.10), we can divide $\tilde{J}$ into four sums:

$$\begin{aligned}\tilde{J} &= \left(\sum_{m,n=\nu-s^*}^{\nu-1} - \sum_{m=\nu-s^*}^{\nu-1} \sum_{n=\nu+1}^{\nu+s^*} - \sum_{m=\nu+1}^{\nu+s^*} \sum_{n=\nu-s^*}^{\nu-1} + \sum_{m,n=\nu+1}^{\nu+s^*} \right) \alpha^{m+n} \\ &= \tilde{J}_1 - \tilde{J}_2 - \tilde{J}_3 + \tilde{J}_4,\end{aligned}$$

where

$$\tilde{J}_1 = \left(\sum_{m=\nu-s^*}^{\nu-1} \alpha^m \right) \left(\sum_{n=\nu-s^*}^{\nu-1} \alpha^n \right) = \left(\frac{\alpha^{\nu-s^*} - \alpha^\nu}{1-\alpha} \right)^2,$$

$$\tilde{J}_4 = \left(\sum_{m=\nu+1}^{\nu+s^*} \alpha^m \right) \left(\sum_{n=\nu+1}^{\nu+s^*} \alpha^n \right) = \left(\frac{\alpha^{\nu+1} - \alpha^{\nu+s^*+1}}{1-\alpha} \right)^2$$

and

$$\tilde{J}_2 = \tilde{J}_3 = \left(\sum_{m=\nu-s^*}^{\nu-1} \alpha^m \right) \left(\sum_{n=\nu+1}^{\nu+s^*} \alpha^n \right) = \left(\frac{\alpha^{\nu-s^*} - \alpha^\nu}{1-\alpha} \right) \left(\frac{\alpha^{\nu+1} - \alpha^{\nu+s^*+1}}{1-\alpha} \right).$$

Therefore,

$$\begin{aligned}\tilde{J} &= \tilde{J}_1 - \tilde{J}_2 - \tilde{J}_3 + \tilde{J}_4 \\ &= \frac{\alpha^{2\nu}}{(1-\alpha)^2}((\alpha^{-s^*} - 1)^2 + \alpha^2(1-\alpha^{s^*})^2 - 2\alpha(\alpha^{-s^*} - 1)(1-\alpha^{s^*})) \\ &= \alpha^{2\nu} \frac{(1-\alpha^{s^*})^2(\alpha^{-s^*} - \alpha)^2}{(1-\alpha)^2} =: \tilde{\sigma}_H^2. \qquad (2.13)\end{aligned}$$

Finally, by (2.10)–(2.13), we get

$$2s^* \left(\frac{1-\alpha^2}{\sigma^2} \right) \mathrm{Var}(W_\nu(s)) = J - \tilde{J} = \sigma_H^2 - \tilde{\sigma}_H^2,$$

where σ_H and $\widetilde{\sigma}_H$ are stated in (2.12) and (2.13), respectively. From this, we deduce that the Haar wavelet transform $W_\nu(s)$ of $\{x_n\}_0^{N-1}$ is distributed as

$$\frac{\sigma\sqrt{\sigma_H^2 - \tilde{\sigma}_H^2}}{\sqrt{2s^*(1-\alpha^2)}} X,$$

where X is a normal distribution with mean 0 and variance 1.

From this, we get the following theorem.

Theorem 2.1. *Let $\{x_n\}_0^{N-1}$ be an $AR(1)$ time series and $W_\nu(s)$ be its Haar wavelet transform. Then the Haar wavelet power spectrum of $\{x_n\}_0^{N-1}$*

$$|W_\nu(s)|^2 \Rightarrow \frac{\sigma^2(\sigma_H^2 - \widetilde{\sigma}_H^2)}{2s^*(1-\alpha^2)} \chi_1^2,$$

where "$\Rightarrow$" indicates "is distributed as" and χ_1^2 is the chi-square distribution with one degree of freedom, $s^* = \frac{s}{2\delta t}$, and

$$\sigma_H^2 - \tilde{\sigma}_H^2 = \frac{2s^*(1-\alpha^2) - 4\alpha(1-\alpha^{s^*}) - 2\alpha^2(1-\alpha^{s^*})^2}{(1-\alpha)^2} - \alpha^{2\nu}\frac{(1-\alpha^{s^*})^2(\alpha^{-s^*}-\alpha)^2}{(1-\alpha)^2}.$$

Remark 2.1. Since $0 \le \alpha < 1$, when ν is sufficiently large such that $\alpha^{2\nu} \approx 0$, the term

$$\tilde{\sigma}_H \approx 0.$$

The above formula is simplified as follows:

$$|W_\nu(s)|^2 \Rightarrow \frac{\sigma^2\sigma_H^2}{2s^*(1-\alpha^2)}\chi_1^2,$$

where σ_H^2 is stated in (2.12).

If $\{x_n\}_0^{N-1}$ is a white noise with variance σ^2, i.e. $\alpha = 0$, then $\sigma_H^2 - \tilde{\sigma}_H^2 = 2s^*$ and

$$|W_\nu(s)|^2 \Rightarrow \sigma^2\chi_1^2.$$

3. Numerical Experiment

The AO is a key aspect of climate variability in the Northern Hemisphere. The AO is defined as the leading empirical orthogonal function (EOF) of Northern Hemisphere

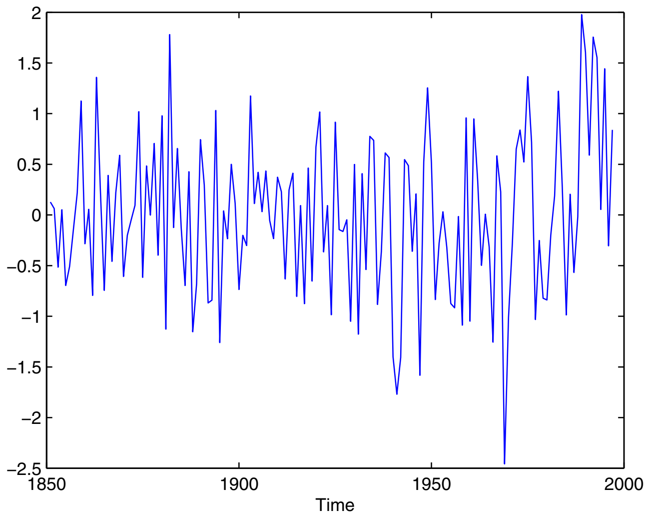

Fig. 1. The standardized time series of winter AO index.

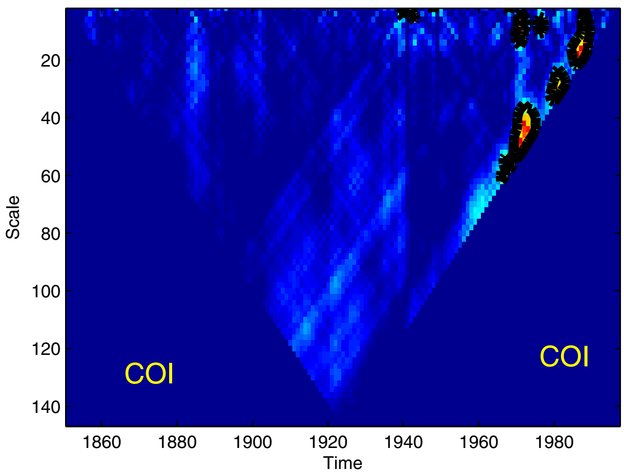

Fig. 2. Haar wavelet power spectrum of AO index. The thick black contour designates the 5% significance level against red noise. In addition, the COI also be marked (color online).

sea level pressure anomalies pole ward of 20°N,[6] and characterized by an exchange of atmospheric mass between the Arctic and middle latitudes. Figure 1 shows the winter AO index (December–February 1851–1997). We will use Haar wavelet to analyze AO index, i.e. we will compute the Haar wavelet power spectra of the AO index. Since the AO index is a finite-length time series, errors in wavelet transform will occur at the beginning and end of the wavelet power spectrum. The Cone of influence (COI) is the region of the wavelet transform in which edge effects become important. Here, we will not research wavelet power spectrum in COI. Using Theorem 2.1, we can extract the instinct information of time series from background red noise. Figure 2 shows Haar wavelet power spectrum of AO index. The thick black contour designates the 5% significance level against red noise. Most regions of significance appear abruptly about 1970. Since the Haar wavelet is very broadband, this discovers a great change in fundamental properties of the AO, commonly called a regime shift or tripping point.

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Appendix A. Haar Wavelet Scale and Fourier Wavelength

In order to give better physical interpretation, one needs to transfer wavelet scale to Fourier wavelength. We will use Meyers' method[5] to obtain the relationship between the equivalent Fourier period T and the wavelet scale a. This relationship estimated by computing the wavelet power spectrum of a cosine wave of a known frequency and then finding the scale a at which the wavelet power spectrum reaches its maximum. In detail:

Let $x(t) = \cos \omega t$. Then

$$W(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} \cos \omega t H\left(\frac{t-b}{a}\right) dt = \sqrt{a} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos \omega (at+b) H(t) dt$$

$$= -\frac{1}{\omega\sqrt{a}} \left(\sin \omega \left(\frac{a}{2}+b\right) + \sin \omega \left(\frac{a}{2}-b\right)\right) = -\frac{2}{\omega\sqrt{a}} \sin \frac{a\omega}{2} \cos b\omega$$

and so

$$|W(a,b)|^2 = \frac{4}{\omega^2 a} \sin^2 \frac{a\omega}{2} \cos^2 b\omega.$$

When ω is such that $\sin \frac{\omega a}{2} = 0$ or $\tan \frac{\omega a}{2} = \omega a$, we have

$$\frac{d}{da} |W(a,b)|^2 = 0.$$

From this, we deduce that the power spectrum $|W_\nu(s)|^2$ attains the maximal value at $a \approx \frac{2.3311}{\omega}$. Note that $\omega = \frac{2\pi}{T}$, where T is the Fourier wavelength. So we obtain the relationship between the scale parameter and the Fourier wavelength as follows:

$$a \approx \frac{2.3311}{2\pi} T = 0.3710T.$$

The relation between scale a and Fourier period T for the Morlet wavelet is $a = 0.97T$.[5] However, for Haar wavelet, the corresponding formula is $a = 0.3710T$. Since for any time series of time step δt and total length $N\delta t$, the range of scales is from the smallest resolvable scale $2\delta t$ to the largest scale $N\delta t$ in wavelet-based time series analysis, by using the Haar wavelet analysis one can extract more low frequency intrinsic information.

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